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LaSalle's Theorem for Stochastic Differential Equations*

HUANG Qingdao (黄庆道)
(*College of Mathematics, Jilin University, Changchun, 130012*)
TIAN Ping (田 萍)
(*College of Commerce, Jilin University, Changchun, 130012*)
WANG Guoming (王国明)
(*College of Mathematics, Jilin University, Changchun, 130012*)

Abstract: In this paper, we establish a the LaSalle's theorem for stochastic differential equation based on Li's work, and give a more general Lyapunov function which it is more easy to apply. Our work has partly generalized Mao's work.

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1 Introduction

It is well known that LaSalle's theorem for locating limit set for nonautonomous systems (see [1], [2]) has become a powerful tool in the study of Lyapunov's stability of differential equations. Li[1] removed the restriction that the direction derivatives of Lyapunov functions remain negative, and make it more convenient to apply those results. With the development of Itô's stochastic calculus, Lyapunov method has been developed to deal with stochastic stability by many authors (see [3]–[9]). However there is a few to study LaSalle's theorem for stochastic differential equations. Mao[3],[4] gave LaSalle's theorems for stochastic differential equation. The main of this paper is to establish a somewhat general version of LaSalle's theorem for stochastic differential equation, based on Mao[3],[4] and Li[1].

2 An Improvement of LaSalle's Theorem

First, let us recall some notations. Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), B_2(t), \cdots B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability

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space. Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$.

we consider the n-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{2.1}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in R^n$. As a standard condition, we impose a hypothesis:

(H_1) Both $f : R^n \times R_+ \to R^n$ and $g : R^n \times R_+ \to R^{nm}$ are measurable function. They satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \cdots$, there is a $c_k > 0$ such that

$$|f(x,t) - f(y,t)| \vee |g(x,t) - g(y,t)| \leq c_k|x - y|$$

for all $t \geq 0$ and $x, y \in R^n$ with $|x| \vee |y| \leq k$, and there is a $c > 0$ such that

$$|f(x,t)| \vee |g(x,t)| \leq c(1 + |x|)$$

for all $(x, t) \in R^n \times R_+$.

It is obvious that under the hypothesis (H_1) the equation (2.1) has a unique continuous solution on $t \geq 0$ (see [5]), which is denoted by $x(t; x_0)$ in this paper. Moreover, for every $p > 0$,

$$E[\sup_{0\leq s\leq t} |x(s; x_0)|^p] < \infty, \qquad t \geq 0.$$

Let $C^{2,1}(R^n \times R_+; R_+)$ denote the family of all nonnegative functions $V(x, t)$ on $(R^n \times R_+)$ which are continuously twice differentiable in x, and once differentiable in t. Define an operator L acting on $C^{2,1}(R^n \times R_+; R_+)$ functions by

$$LV(x.t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\text{trace}[g(x,t)^T V_{xx} g(x,t)],$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, \quad V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_n}\right),$$

$$V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Moreover, we denote by $L^1(R_+, R_+)$ the family of all functions $b : R_+ \to R_+$ such that

$$\int_0^\infty b(t)dt < \infty.$$

We can now give our stochastic version of LaSalle's theorem as follows.

Theorem 2.1 *Let (H_1) hold. Assume that there is a function $V(x,t) \in C^{2,1}(R^n \times R_+, R)$, functions $a(t)$, $b(t) \in L^1(R_+, R_+)$, and a continuous function $w : R^n \to R_+$ such that*

$$\lim_{|x|\to\infty} \inf_{0\leq t<\infty} V(x,t) = \infty \tag{2.2}$$

and

$$LV(x,t) \leq b(t) - w(x) + a(t)V(x,t). \tag{2.3}$$

Moreover, for each initial value $x_0 \in R^n$ there is a $p > 2$ such that

$$\sup_{0\leq t<\infty} E|x(t,x_0)|^p < \infty. \tag{2.4}$$

Then, for every $x_0 \in R^n$, $\lim_{t\to\infty} V(x,t)$ exists and is finite almost surely, and moreover,

$$\lim_{t\to\infty} w(x(t,x_0)) = 0 \ a.s. \tag{2.5}$$

Remark In the case of $a(t) \equiv 0$ our result is Mao's result in [3].

In order to prove Theorem 2.1, we need the following useful lemmas.

Lemma 2.1 *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0$ -measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \qquad for\ t \geq 0.$$

If $X(t)$ is nonnegative, then

$$\{\lim_{t\to\infty} A(t)\} \subset \{\lim_{t\to\infty} X(t) exists\ and\ is\ finite\} \cap \{\lim_{t\to\infty} U(t) < \infty\},$$

where $B \subset D$ a.s. means $P(B \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s., then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X(t,\omega)\ exists\ and\ is\ finite,\ and\ \lim_{t\to\infty} U(t,\omega) < \infty.$$

This lemma is established by Liptser and Shiryayev (see [10], Theorem 7, p.139). The next lemma is the well-known Kolmogorov-Čentsov theorem on the continuity of a stochastic process derived from the moment property.

Lemma 2.2 *Suppose that an n-dimensional stochastic process $X(t)$ on $t \geq 0$ satisfies the condition*

$$E|X(t) - X(s)|^\alpha \leq C|t-s|^{1+\beta}, \qquad 0 \leq s,t < \infty$$

for some positive constants α, β and C. Then there exists a continuous modification $\tilde{X}(t)$ of $X(t)$, which has the property that for every $\gamma \in (0, \beta/\alpha)$, there is a positive random variable $h(\omega)$ such that

$$P\left\{\omega : \sup_{\substack{0<t-s<h(\omega)\\ 0\leq s,t<\infty}} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t-s|^\gamma} \leq \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ is locally but uniformly Hölder- continuous with exponent γ.

The proof of this result can be found in [11] in the case when the stochastic process $X(t)$ is on the finite interval $[0,T]$, but a little bit of modification of the proof is needed for the case when $X(t)$ is on the entire R_+.

Lemma 2.3 *Let (H_1) and (2.4) hold. Set*

$$y(t) := \int_0^t g(x(s),s)dB(s) \qquad on\ t \geq 0,$$

where we write $x(t,x_0) = x(t)$ simply. Then almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$.

The proof of this result can be found in [3].

Now we give the proof of Theorem 2.1.

Proof. Fix any initial value x_0 and write $x(t,x_0) = x(t)$ simply. Let

$$Z(t) = \exp\left\{-\int_0^t a(s)ds\right\}.$$

Applying Itô formula on $Z(t)V(x(t),t)$ and condition (2.3), we have

$$\begin{aligned} Z(t)V(x(t),t) &= V(x_0,0)+\int_0^t L(Z(s)V(x(s),s))ds \\ &\quad +\int_0^t (Z(s)V(x(s),s))_x g(x(s),s)dB(s) \\ &= V(x_0,0)+\int_0^t Z(s)(LV(x(s),s)-a(s)V(x(s),s))ds \\ &\quad +\int_0^t Z(s)V_x(x(s),s)g(x(s),s)dB(s) \\ &\le V(x_0,0)+\int_0^t Z(s)b(s)ds-\int_0^t Z(s)w(x(s))ds \\ &\quad +\int_0^t Z(s)V_x(x(s),s)g(x(s),s)dB(s). \end{aligned}$$

Since

$$\int_0^\infty b(s)ds<\infty, \qquad \lim_{t\to\infty} Z(t)<\infty,$$

we have

$$\int_0^\infty Z(s)b(s)ds<\infty, \qquad Z(s)w(x(s))\ge 0.$$

By Lemma 2.1, we obtain that for almost every $\omega\in\Omega$,

$$\int_0^\infty w(x(t,\omega))dt<\infty \tag{2.6}$$

and

$$\lim_{t\to\infty} V(x(t,\omega),t) \text{ exists and is finite.} \tag{2.7}$$

We first claim that almost every sample path of $x(t)$ is uniformly continuous on $t\ge 0$. We write $x(t)=x_0+z(t)+y(t)$, where

$$z(t)=\int_0^t f(x(s),s)ds \qquad \text{and} \qquad y(t)=\int_0^t g(x(s),s)dB(s).$$

By Lemma 2.3, (2.6)–(2.7) and $\lim_{t\to\infty} Z(t)<\infty$, we see that there is an $\bar\Omega\subset\Omega$ with $P(\bar\Omega)=1$ such that for every $\omega\in\bar\Omega$ (2.6) and (2.7) hold; moreover, $y(t,\omega)$ is uniformly continuous on $t\ge 0$. Now, fix any $\omega\in\bar\Omega$. By (2.7),

$$\sup_{0\le t<\infty} V(x(t,\omega),t)<\infty.$$

Hence, by (2.2), there is a positive number $h(\omega)$ such that

$$|x(t,\omega)|\le h(\omega) \qquad \text{for all } t\ge 0.$$

From this and the hypothesis (H_1) we get that for $0\le s<t<\infty$,

$$\begin{aligned} |z(t,\omega)-z(s,\omega)| &\le \int_s^t |f(x(r,\omega),r)|dr \\ &\le c\int_s^t (1+|x(r,\omega)|)dr\le c(1+h(\omega))(t-s), \end{aligned}$$

which implies that $z(t,\omega)$ is uniformly continuous on $t\ge 0$. Since $\omega\in\bar\Omega$ is arbitrary, we have proved that for every $\omega\in\bar\Omega$, $x(t,\omega)$ is uniformly continuous on $t\ge 0$.

We next claim that

$$\lim_{t\to\infty} w(x(t,\omega))=0 \qquad \text{for all } \omega\in\bar\Omega. \tag{2.8}$$

If this is not true, then for some $\hat{\omega} \in \bar{\Omega}$

$$\limsup_{t\to\infty} w(x(t,\hat{\omega})) > 0.$$

So there is some $\varepsilon > 0$ and a sequence $\{t_k\}_{k\geq 1}$ of positive numbers with $t_k + 1 < t_{k+1}$ such that

$$w(x(t_k,\bar{\omega})) > \varepsilon \qquad \text{for all } k \geq 1. \tag{2.9}$$

Set $\bar{S}_h = \{x \in R^n : |x| \leq h\}$, where $h = h(\hat{\omega})$ has been defined above in the way that $\{x(t,\bar{\omega}) : t \geq 0\} \subset \bar{S}_h$. Since it is continuous, $w(\cdot)$ must be uniformly continuous in $\bar{S}_h$ and there is a $\delta_1 > 0$ such that

$$|w(x) - w(y)| < \frac{\varepsilon}{2} \qquad \text{if } x, y \in \bar{S}_h, |x-y| < \delta_1. \tag{2.10}$$

On the other hand, recalling that $x(t,\hat{\omega})$ is uniformly continuous on $t \geq 0$, we can find a $\delta_2 \in (0,1)$ such that

$$|x(t,\hat{\omega}) - x(s,\hat{\omega})| < \delta_1 \quad \text{if } 0 \leq s,t < \infty, |t-s| \leq \delta_2. \tag{2.11}$$

Combining (2.10) and (2.11), we see that for every $n > 1$,

$$|w(x(t_k,\hat{\omega})) - w(x(t,\hat{\omega}))| < \frac{\varepsilon}{2} \qquad \text{if } t_k \leq t \leq t_k + \delta_2.$$

This, together with (2.9), yields

$$\begin{aligned} w(x(t,\hat{\omega})) &\geq w(x(t_k,\hat{\omega})) - |w(x(t_k,\hat{\omega})) - w(x(t,\hat{\omega}))| \\ &> \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty w(x(t,\hat{\omega})dt &\geq \sum_{k=1}^\infty \int_{t_k}^{t_k+\delta_2} w(x(t,\hat{\omega}))dt \\ &\geq \sum_{k=1}^\infty \frac{\varepsilon\delta_2}{2} = \infty, \end{aligned}$$

which contradicts (2.6) since we have already shown that (2.6) holds for all $\omega \in \bar{\Omega}$ and of course for $\hat{\omega}$. Hence, (2.8) must be true and the theorem has been proved.

As an application, we give an example to which our LaSalle's result apply and [3] does not.

Example Consider a one-dimensional stochastic differential equation

$$dx(t) = \frac{1}{2}(\exp\{-t\} - 2)xdt + xdB(t), \qquad t \geq 0. \tag{2.12}$$

Let

$$V(x,t) = x^2, \quad w(x) = x^2, \quad a(t) = \exp\{-t\}, \quad b(t) = 0.$$

Then

$$LV(x,t) = -x^2 + \exp\{-t\}x^2 = -w(x) + a(t)V(x,t).$$

The conditions of Theorem 2.1 hold, and so for any initial value $x_0 \in R^n$

$$\lim_{t\to\infty} x(t,x_0) = 0.$$

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