Linear Tests for DARA Stochastic Dominance[1]

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Abstract: We develop and implement linear formulations of convex stochastic dominance (SD) relations based on decreasing absolute risk aversion (DARA) for discrete and polyhedral choice sets. Our approach is based on a piecewise-exponential representation of utility and a local linear approximation to the exponentiation of log marginal utility. An empirical application to historical stock market data suggests that a passive stock market portfolio is DARA SD inefficient relative to concentrated portfolios of small-cap stocks. The mean-variance rule and N-th order stochastic dominance rules substantially underestimate the degree of market portfolio inefficiency, because they do not penalize the unfavorable skewness of diversified portfolios, in violation of DARA.

Key words: Stochastic dominance, utility theory, decreasing absolute risk aversion, linear programming, bootstrapping, market portfolio efficiency, pricing kernel, skewness.

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1. Introduction

Decreasing absolute risk aversion stochastic dominance (DSD), introduced by Vickson (1975a, 1975b, 1977)), is one of the most appealing stochastic dominance (SD) criteria. DSD assumes non-satiation, risk aversion and decreasing absolute risk aversion (DARA) without imposing a specific functional form for the utility function. These assumptions are widely accepted as minimum regularity conditions in utility theory; for example, they are necessary conditions for 'proper risk aversion' (Pratt and Zeckhauser, 1987) and 'standard risk aversion' (Kimball, 1993). In many applications, relaxing these assumptions would result in a substantial loss of discriminating power.

Despite its theoretical appeal, DSD has proven to be relatively difficult to implement in practice. The analytic complexity of DSD stems from restricting the ratio of the utility function's curvature (second-order derivative) to its slope (first-order derivative). Even for the simple case of pairwise comparison between two prospects, a closed-form solution does not exist and numerical optimization is required to verify whether one prospect DSD dominates another prospect. Vickson (1975a) formulated *pairwise DSD comparison* as a non-linear problem that can be solved through a dynamic programming routine.

To the best of our knowledge, a DSD algorithm does not exist for the more general case of *convex stochastic dominance*, which compares a given prospect with a set of multiple alternatives rather than a single alternative. Fishburn (1974) shows that multiple pairwise comparisons generally cannot detect all non-optimal prospects. While Fishburn's original analysis considered the choice from a finite number of prospects, Shalit and Yitzhaki (1994), among others, assume that linear combinations of the individual prospects are feasible, for example, mixtures of production methods, marketing instruments or financial securities.

Third-order stochastic dominance (TSD; Whitmore, 1970) is often used as a more computationally friendly surrogate for DSD. TSD replaces DARA with the assumption of prudence (the third-order derivative is non-negative). For pairwise TSD comparison, a well-known closed-form solution exists based on comparing the third-order integrated distribution functions and the means of the two prospects in question. For convex TSD, Bawa *et al*. (1985) provide linear programming (LP) formulations.

Unfortunately, the assumption of prudence is weaker than the assumption of DARA, leading to a possible loss of discriminating power. For example, TSD (and every higher-order SD criterion) allows utility to be quadratic (increasing absolute risk aversion; IARA) for the entire range where utility is increasing. Hence, TSD inherits the known weaknesses of quadratic utility and mean-variance (M-V) analysis. In addition, Basso and Pianca (1997) point out that, with regard to the problem of determining lower and upper bounds for the price of a financial option contract, the DARA rule improves the stochastic dominance criteria of any order.

This study develops linear formulations for convex DSD comparisons. To arrive at finite optimization problems, we focus on discrete probability distributions. Empirical studies generally use discrete sample distributions, and experimental studies generally use prospects with a discrete population distribution. In addition, many continuous population distributions can be approximated accurately with some discrete distribution, for example, by means of a finite number of random draws from the population distribution.

Our approach is based on a piecewise-exponential representation of DARA utility functions and a (tight) local linear approximation to the exponentiation of log marginal utility. This approach applies generally for comparing a given prospect with a discrete set of alternative prospects (for instance, pairwise comparison) and for comparison with a polyhedral set of linear combinations of prospects. The appendix provides an extension of our framework to impose increasing relative risk aversion (IRRA) using piecewise-power utility functions.

Our approach can be implemented by solving a relatively small system of linear inequalities. The LP approach to SD seems particularly convenient for reducing the computational burden of statistical resampling methods, which analyze not only the original sample but also thousands of random pseudo-samples or sub-samples drawn from the original sample. These methods have emerged as the dominant method for statistical inference in empirical applications of SD since the pioneering work by Nelson and Pope (1991).

The empirical part of our study applies the DSD rule and several other decision criteria to analyze the efficiency of a broad stock market portfolio relative to alternative portfolios formed from a set of benchmark assets using historical return data. The analysis is relevant because several capital market equilibrium models predict that the market portfolio is efficient. Another reason for expecting market portfolio efficiency is the popularity of passive mutual funds and exchange traded funds that passively track broad stock market indices. The empirical analysis shows that the pricing errors of small-cap stocks critically depend on imposing DARA rather than prudence. The application also illustrates the goodness of the linear approximation to the DSD criterion.

2. Preliminaries

We consider M distinct prospects with risky outcomes, $\tilde{x}_1, \cdots, \tilde{x}_M \in \mathbb{R}$. The outcomes are treated as random variables with a discrete, state-dependent, joint probability distribution characterized by R mutually exclusive and exhaustive scenarios with probabilities $p_r > 0, r = 1, \cdots, R$. We use $x_{i,r}$ for the realized outcome of prospect i in scenario r; collect all possible outcomes across prospects and states in $\mathcal{Y} := \{y \in \mathbb{R}: y = x_{i,r} \quad i = 1, \cdots, M; r = 1, \cdots, R\}$; rank these values in ascending order $y_1 \leq \cdots \leq y_S$; and use $q_{i,s} := \mathbb{P}[\tilde{x}_i = y_s] = \sum_{r: x_{i,r} = y_s} p_r$, $i = 1, \cdots, M$, $s = 1, \cdots, S$.

Decision makers' preferences are described by three times continuously differentiable, von Neumann-Morgenstern utility functions $u(x): \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \coloneqq [y_1, y_S]$. Using $a(x) := -u''(x)/u'(x)$ for the Arrow-Pratt absolute risk aversion (ARA) quotient, the DSD functions can be represented as follows:

$$\mathcal{U}_3^* := \{u \in \mathcal{U}_3: u'(x) > 0; a'(x) \leq 0 \quad \forall x \in \mathcal{D}\}; \tag{1}$$

$$\mathcal{U}_3 := \{u \in \mathcal{C}^3: u'(x) \geq 0, u''(x) \leq 0, u'''(x) \geq 0 \quad \forall x \in \mathcal{D}\}, \tag{2}$$

where $\mathcal{U}_3$ represents the TSD functions.

Continuous differentiability is assumed to simplify the notation and economic interpretation, and can be relaxed using super-differential calculus for general concave functions. Indeed, our analysis below will sometimes represent utility and marginal utility using piecewise-polynomial and piecewise-exponential functions. The restrictions on the derivatives of the TSD functions (2) have the economic meaning of non-satiation ($u'(x) \geq 0 \ \forall x \in \mathcal{D}$), risk aversion ($u''(x) \leq 0 \ \forall x \in \mathcal{D}$) and prudence ($u'''(x) \geq$

$0\ \forall x \in \mathcal{D}$). The DSD functions (1) impose the additional restriction of DARA ($a'(x) \leq 0\ \forall x \in \mathcal{D}$). This restriction makes the prudence restriction redundant, because $a'(x) \leq 0 \Rightarrow u'''(x) \geq 0$, and, in addition, requires strict (rather than weak) monotonicity ($u'(x) > 0\ \forall x \in \mathcal{D}$), because marginal utility enters as the divisor in the ARA quotient.

We distinguish between two types of SD relations: one involving the choice from a discrete set of prospects and another for the choice from a convex set of prospects. In both cases, a given feasible prospect is evaluated relative to all feasible prospects.

DEFINITION 1 (OPTIMALITY) *A given prospect $\tilde{x}_i$, $i \in \{1, \cdots, M\}$, is optimal in terms of DSD (TSD) relative to the set of prospects $\{\tilde{x}_1, \cdots, \tilde{x}_M\}$ if there exists an admissible utility function $u \in \mathcal{U}_3^*$ ($u \in \mathcal{U}_3$) for which it is preferred to every alternative prospect:*

$$\sum_{s=1}^{S} u(y_s)\left(q_{i,s} - q_{j,s}\right) \geq 0, \qquad j = 1, \cdots, M. \tag{3}$$

If the choice set consists of two prospects ($M = 2$), then the DSD optimality criterion is equivalent to Vickson's (1975a) pairwise DSD criterion, and can be implemented using a dynamic programming algorithm that treats the ratios $\left(u(y_{s+1}) - u(y_s)\right)/\left(u(y_s) - u(y_{s-1})\right)$, $s = 2, \cdots, S-1$, as model variables. If more choices are available ($M > 2$), then an optimality test generally is more powerful than 'simply' performing $(M-1)$ pairwise dominance tests, because the evaluated prospect may be non-optimal for every admissible utility function even if it is not dominated by any of the alternative prospects.

In addition to analyzing a finite number of prospects, our analysis also considers the case where convex combinations of the prospects are feasible, following Shalit and Yitzhaki (1994), Post (2003) and Kuosmanen (2004). For this case, we represent the choice set by:

$$\mathcal{X} := \left\{ \sum_{j=1}^{M} \lambda_j \tilde{x}_j : \sum_{j=1}^{M} \lambda_j = 1;\ \lambda_j \geq 0, \qquad j = 1, \ldots, M \right\}. \tag{4}$$

Post (2003, Section I) and Post and Versijp (2007, Section IV) demonstrate how the analysis can be generalized to a general polyhedral choice set of linear combinations of prospects under linear constraints, using either a vertex representation or a halfspace representation.

Our analysis allows for the inclusion of a riskless prospect along the lines of Levy and Kroll (1978). Including a riskless asset can sometimes simplify the analysis by considering only choice alternatives (mixtures of risky prospects and the riskless prospect) that have the same mean as the evaluated prospect. This approach could be particularly interesting for testing DSD efficiency because the TSD and DSD rules are equivalent for prospects with the same mean (Vickson (1975b, Thm 4, p. 805)). Loosely speaking, the relation between the preferences over the *shape* of the outcomes distribution and the *location* of the distribution is inconsequential for comparing distributions with the same mean. Unfortunately, this approach generally does not apply if restrictions are imposed on the riskless prospect. In addition, the dominance relation between prospects with equal means is generally not robust to small data perturbations, which complicates empirical tests.

We use $\tilde{x}^e \in \mathcal{X}$ for the evaluated prospect. The evaluated prospect may be a corner solution, that is, some non-negativity constraints may be binding ($\lambda_j = 0$). The ordering of the scenarios is inconsequential in our analysis and we are free to label the scenarios by their ranking with respect to the evaluated prospect: $x_1^e \leq \cdots \leq x_R^e$. We stress that a different choice of evaluated combination generally involves a different ranking.

DEFINITION 2 (EFFICIENCY) *A given prospect $\tilde{x}^e \in \mathcal{X}$ is efficient in terms of DSD (TSD) relative to all feasible prospects $\tilde{x} \in \mathcal{X}$ if it is the optimum for some admissible utility function $u \in \mathcal{U}_3^*$ ($u \in \mathcal{U}_3$):*

$$\sum_{r=1}^{R} p_r u'(x_r^e)\left(x_r^e - x_{j,r}\right) \geq 0, \qquad j = 1, \cdots, M. \tag{5}$$

This definition follows from the Karush-Kuhn-Tucker first-order conditions for the utility optimization problem $\max_{x \in \mathcal{X}} \sum_{r=1}^{R} p_r u(x_r)$: a marginal adjustment to the optimal weight of any given prospect should result in a reduction of expected utility compared with the optimal solution. The usual complementary slackness condition applies: the inequalities in (5) are always binding for prospects that are included in the evaluated prospect ($\lambda_j > 0$), but the inequalities may be non-binding for 'inactive prospects' ($\lambda_j = 0$).

To allow for a compact and general notation, the analysis below will use $T = R, S$ to mean the number of scenarios R or the number of possible outcomes S, depending on the context. Similarly, we will use $t = r, s$ and $z_t = x_r^e, y_s$, depending on the context.

To avoid numerical instability, we use the (data-dependent) normalization $u'(z_m) = 1$, where z_m is the median outcome of the evaluated prospect, that is, $m \coloneqq \min_t\{t: \mathbb{P}(\tilde{z} \leq z_t) \geq 0.5\}$. The advantage of this particular normalization is that the relevant outcome level z_m is not affected by a monotone transformation of marginal utility. If desired, we can always rescale marginal utility after the analysis, for example, to achieve an average value of unity (the preferred normalization for our asset pricing application).

3. Piecewise-exponential utility

We can formulate DSD utility (1) in terms of log marginal utility $\ell(x) := \ln(u'(x))$, which is the negative anti-derivative of the ARA quotient: $-\ell'(x) = a(x)$. DARA requires $a'(x) \leq 0 \Leftrightarrow \ell''(x) \geq 0$, or, equivalently, $u'(x)$ is log-convex. This insight motivates the following formulation in terms of second-order stochastic dominance (SSD):

PROPOSITION 1 (EXPONENTIATION)

$$\mathcal{U}_3^* = \{u \in \mathcal{U}_3 : u'(x) = \exp(\ell(x)) \quad -\ell \in \mathcal{U}_2 \quad \forall x \in \mathcal{D}\}; \tag{6.1}$$

$$\mathcal{U}_2 \coloneqq \{u \in \mathcal{C}^2 : u'(x) \geq 0, u''(x) \leq 0 \quad \forall x \in \mathcal{D}\}. \tag{6.2}$$

(The proofs of our propositions and theorems should be evident from the preceding and subsequent discussions and are therefore not provided separately.)

The DSD criterion thus imposes the same structure (monotonicity and concavity) for negative log marginal utility that the SSD criterion imposes for utility ($\mathcal{U}_2$). Post (2003) showed that the SSD criterion can be formulated in terms of piecewise-linear utility functions that are constructed via summation by parts. We can use the same approach to linearize the levels of log marginal utility ($\ell(x)$) for DSD functions:

PROPOSITION 2 (DSD LOG MARGINAL UTILITY) *For any (normalized) utility function* $u \in \mathcal{U}_3^*$ *and a discrete set of outcomes* $z_1 \leq \cdots \leq z_T$, *we can represent the levels of log marginal utility by the corresponding levels of a decreasing and convex piecewise-linear function of the outcome levels:*

$$\ell(z_t) = \sum_{k=t}^{T-1} \rho_k (z_{k+1} - z_t) + \rho_T, \qquad t = 1, \cdots, m-1, m+1, \cdots, T; \tag{7.1}$$

$$\ell(z_m) = \sum_{k=m}^{T-1} \rho_k (z_{k+1} - z_m) + \rho_T = 0; \tag{7.2}$$

$$\rho_t \geq 0, \qquad t = 1, \cdots, T-1. \tag{7.3}$$

The log marginal utility levels (7.1) are built from the ARA decrements $\rho_t = a(\bar{z}_t) - a(\bar{z}_{t+1}), t = 1, \cdots, T-2$, and the ARA level $\rho_{T-1} = a(\bar{z}_{T-1})$, for some tangency points $\bar{z}_t \in [z_t, z_{t+1}]$, $t = 1, \cdots, T-1$, and $\rho_T = \ell(z_T)$, using the first Mean Value Theorem for integration. Restriction (7.2) is a normalization of median log marginal utility that follows from our normalization of marginal utility: $u'(z_m) = 1$. The non-negativity constraints (7.3) impose DARA. The sign of ρ_T is not explicitly restricted, but normalization (7.2) implies $\rho_T \leq 0$.

The piecewise-linear structure of log marginal utility implies that the DSD criterion can be represented by piecewise-exponential functions that are obtained by means of integration over the exponentiation of piecewise-linear log marginal utility $\ell(x)$:

$$u(z_t) = \begin{cases} -\left(\sum_{k=t}^{T-1} \rho_k\right)^{-1} \exp\left(\sum_{k=t}^{T-1} \rho_k (z_{k+1} - z_t) + \rho_T\right) + \varphi_t, & t = 1, \ldots, T_0 - 1 \\ \exp(\rho_T) z_t, & t = T_0, \ldots, T; \end{cases} \tag{7.4}$$

$$T_0 := \min_t \left\{ t : \sum_{k=t}^{T} \rho_k = \rho_T \right\}. \tag{7.5}$$

The constants $\varphi_t, t = 1, \ldots, T_0 - 1$, are selected to ensure continuity of the utility levels at the interval boundaries: $\lim_{x \to (\bar{z}_t)} u(x) = u(\bar{z}_t)$, $t = 1, \ldots, T_0 - 1$.

Placing the piecewise-exponential function (7.4)-(7.5) in our Definition 1 or Definition 2 yields necessary and sufficient conditions for DSD optimality and DSD efficiency of finite dimensions. Unfortunately, the resulting formulations are non-linear and generally non-convex, which makes this approach unpractical. The next section therefore introduces a useful linear approximation.

4. Linear DARA SD restrictions

Our strategy is to linearize $u \in \mathcal{U}_3$ and the exponentiation $u'(x) = \exp(\ell(x))$ (in addition to the linearization of $\ell(z_t)$ in Proposition 2). We use an exact linearization for $u \in \mathcal{U}_3$ and a (tight) local linear approximation for $u'(x) = \exp(\ell(x))$.

Post and Kopa (2013, Thm 1) represent the general N-th order SD criterion (which does not cover DSD) by using general piecewise-polynomial functions that are linear in the parameters. Applying their analysis to the special case of TSD ($N = 3$), and using our normalization, we find:

PROPOSITION 3 (TSD UTILITY) *For any (normalized) utility function $u \in \mathcal{U}_3$, and a discrete set of outcomes $z_1 \leq \cdots \leq z_T$, we can represent the levels of utility (marginal utility) by the corresponding levels of an increasing and concave piecewise-quadratic (decreasing and convex piecewise-linear) function of the outcome levels:*

$$u(z_t) = \frac{1}{2}\sum_{k=t}^{T-1} -\gamma_k(z_{k+1}-z_t)^2 + \gamma_T(z_t - z_T) + \gamma_{T+1}, \qquad t = 1,\cdots,T; \tag{8.1}$$

$$u'(z_t) = \sum_{k=t}^{T-1} \gamma_k(z_{k+1}-z_t) + \gamma_T, \qquad t = 1,\cdots,m-1; m+1,\cdots,T; \tag{8.2}$$

$$u'(z_m) = \sum_{k=m}^{T-1} \gamma_k(z_{k+1}-z_m) + \gamma_T = 1; \tag{8.3}$$

$$\gamma_t \geq 0, \qquad t = 1,\cdots,T. \tag{8.4}$$

The utility levels (8.1) and marginal utility levels (8.2) are built from the increments to the second-order derivative $\gamma_t = u''(\bar{\bar{z}}_{t+1}) - u''(\bar{\bar{z}}_t)$, $t = 1,\cdots,T-2$, $\gamma_{T-1} = -u''(\bar{\bar{z}}_{T-1})$, for some tangency points $\bar{\bar{z}}_t \in [z_t, z_{t+1}]$, $t = 1,\cdots,T-1$, and $\gamma_T = u'(z_T)$, $\gamma_{T+1} = u(z_T)$, by a Taylor expansion. Restriction (8.3) is the normalization of median marginal utility. The non-negativity constraints (8.4) impose the regularity conditions: non-satiation ($\gamma_T \geq 0$), risk aversion ($\gamma_{T-1} \geq 0$) and prudence ($\gamma_t \geq 0 \quad t = 1, \cdots, T-2$).

We have now derived two sets of linear conditions: (7) expresses negative log marginal utility as a linear function of the levels and changes of the ARA quotient; (8) expresses utility and marginal utility as linear functions of the levels and changes of the second-order derivative. The relation between these two sets of restrictions is non-linear: $u'(z_t) = \exp\big(\ell(z_t)\big)$. We will therefore develop a set of linear conditions based on a local first-order approximation to the exponentiation. It is insightful to first consider the general polynomial approximation.

Let c_t be a prior estimate for $u'(z_t)$, the specification of which will be discussed below, and $\ln(c_t)$ the associated estimate for $\ell(z_t)$, $t = 1, \cdots, T$. The estimates are assumed to be positive, monotonic and normalized: $c_1 \geq \cdots \geq c_T > 0$, $c_m = 1$. We may apply a local K-th order Taylor series approximation of the exponential function $g(x) = \exp(x)$ at point $\ell(z_t)$ around point $\ln(c_t)$, $t = 1, \cdots, T$:

$$\begin{aligned}\hat{g}_K\big(\ell(z_t)\big) &\coloneqq g(\ln(c_t)) + \sum_{k=1}^{K} \frac{1}{k!} \frac{\partial^k g(\ln(c_t))}{\partial \ln(c_t)^k} (\ell(z_t) - \ln(c_t))^k \\ &= \exp(\ln(c_t)) + \sum_{k=1}^{K} \frac{1}{k!} \exp(\ln(c_t))(\ell(z_t) - \ln(c_t))^k \\ &= c_t \left(1 + \sum_{k=1}^{K} \frac{1}{k!} (\ell(z_t) - \ln(c_t))^k \right).\end{aligned} \tag{9}$$

The linear approximation ($K = 1$) is of particular interest for our purposes:

LEMMA 1 (EXPONENTIATION) *For any utility function $u \in \mathcal{U}_3^*$, a discrete set of outcomes $z_1 \leq \cdots \leq z_T$, and prior marginal utility estimates $c_1 \geq \cdots \geq c_T > 0$, $c_m = 1$, the following linear inequalities apply:*

$$u'(z_t) \geq \hat{g}_1\big(\ell(z_t)\big) = c_t(1 + \ell(z_t) - \ln(c_t)), \qquad t = 1, \cdots, T. \tag{10}$$

We stress that we do not approximate the utility function with an exponential function (or any other particular functional form) and we also do not use a Taylor series approximation to the utility function. Rather, we use a local linear approximation for the exponentiation of log marginal utility in order to linearize the non-linear DARA condition.

The goodness of the approximation depends on the specification of the prior estimates c_t for the marginal utility levels $u'(z_t)$, $t = 1, \cdots, T$. Our preferred approach uses a 'frame function': a parametric utility function $v \in \mathcal{U}_3^*$ (for example, a constant relative risk aversion (CRRA) power function) that obeys our normalization ($v'(z_m) = 1$) and is calibrated to the decision problem in question. By setting $c_t = v'(z_t),\ t = 1, \cdots, T$, the prior estimates have the same normalization, general level of risk aversion and general properties (positivity, monotonicity, log-convexity) as the marginal utility functions that we seek to approximate. The numerical example and empirical application below further illustrate the use of the frame function.

We use (6), (7), (8) and (10) to derive the following linear conditions for DSD optimality:

THEOREM 1 (OPTIMALITY) *A necessary condition for DSD optimality of a given prospect* $\tilde{x}_i$, $i \in \{1, \cdots, M\}$, *is that, for any given set of prior marginal utility estimates* $c_1 \geq \cdots \geq c_S > 0$, $c_m = 1$, *there exists a solution* γ_s^*, $s = 1, \cdots, S+1$; ρ_s^*, $s = 1, \cdots, S$, *for the following system of linear inequalities*:

$$\sum_{s=1}^{S}\left(\frac{1}{2}\sum_{k=s}^{S-1} -\gamma_k(y_{k+1} - y_s)^2 + \gamma_S(y_s - y_S) + \gamma_{S+1}\right)\left(q_{i,s} - q_{j,s}\right) \geq 0, \qquad j = 1, \cdots, M; \tag{11.1}$$

$$\sum_{k=s}^{S-1} \gamma_k(y_{k+1} - y_s) + \gamma_S \geq c_s\left(1 + \sum_{k=s}^{S-1} \rho_k(y_{k+1} - y_s) + \rho_S - \ln(c_s)\right), \qquad s = 1, \cdots, S; \tag{11.2}$$

$$\sum_{k=m}^{S-1} \gamma_k(y_{k+1} - y_m) + \gamma_S = 1; \tag{11.3}$$

$$\sum_{k=m}^{S-1} \rho_k(y_{k+1} - y_m) + \rho_S = 0; \tag{11.4}$$

$$\gamma_s \geq 0, \qquad s = 1, \cdots, S; \tag{11.5}$$

$$\rho_s \geq 0, \qquad s = 1, \cdots, S-1. \tag{11.6}$$

A sufficient condition for DSD optimality is that some feasible solution γ_s^*, $s = 1, \cdots, S+1$*;* ρ_s^*, $s = 1, \cdots, S$, *exhibits a log-convex pattern*:

$$\ln\left(\sum_{k=s}^{S-1} \gamma_k^*(y_{k+1} - y_s) + \gamma_S^*\right) = \sum_{k=s}^{S-1} \rho_k^*(y_{k+1} - y_s) + \rho_S^*,$$
$$s = 1, \cdots, m-1, m+1, \cdots, S. \tag{12}$$

The utility function is quadratic in the outcome levels but linear in parameters. To illustrate the economic meaning of the parameters and prove the necessary condition, suppose that the evaluated prospect is preferred to every alternative prospect for $u \in \mathcal{U}_3^*$. The same preference relation will then apply for the normalized function $v = u/u'(y_m)$, $v \in \mathcal{U}_3^*$. A solution to system (11) then is $\rho_s = a(\bar{y}_s) - a(\bar{y}_{s+1})$, $s = 1, \cdots, S-2$, and $\rho_{S-1} = a(\bar{y}_{S-1})$, for some tangency points $\bar{y}_s \in [y_s, y_{s+1}]$, $s = 1, \cdots, S-1$; $\rho_S = \ell(y_S)$; $\gamma_s = v''(\bar{\bar{y}}_{s+1}) - v''(\bar{\bar{y}}_s)$, $s = 1, \cdots, S-2$, $\gamma_{S-1} = -v''(\bar{\bar{y}}_{S-1})$, for some tangency points $\bar{\bar{y}}_s \in [y_s, y_{s+1}]$, $s = 1, \cdots, S-1$; $\gamma_S = v'(y_S)$; and $\gamma_{S+1} = v(y_S)$.

The system (11) gives a necessary but not sufficient condition, because some feasible solutions may not obey DARA due to approximation error for the exponentiation of log marginal utility in (11.2). If we could somehow identify a solution that does obey DARA, or log-convex marginal utility, then that would suffice to prove optimality. In this case, the solution would also obey system (11), but without approximation error, if we set $c_s = \sum_{k=s}^{S-1} \gamma_k^*(y_{k+1} - y_s) + \gamma_S^*$, $s = 1, \cdots, S$. The sufficient condition (12) unfortunately becomes non-linear if we have to search for a feasible solution. The primary purpose of the sufficient condition therefore is to diagnose solutions *a posteriori*, that is, after testing the necessary condition, as we will see below.

Similarly, we can derive the following linear conditions for DSD efficiency:

THEOREM 2 (EFFICIENCY) A *necessary condition for DSD efficiency of a given prospect* $\tilde{x}^e \in \mathcal{X}$ *is that, for any given set of prior marginal utility estimates* $c_1 \geq \cdots \geq c_R > 0$, $c_m = 1$, *there exists a solution* γ_r^*, $r = 1, \cdots, R$*;* ρ_r^*, $r = 1, \cdots, R$, *for the following system of linear inequalities*:

$$\sum_{r=1}^{R}\left(\sum_{k=r}^{R-1}\gamma_k(x_{k+1}^e - x_r^e) + \gamma_R\right)(x_r^e - x_{j,r})\, p_r \geq 0, \qquad j = 1,\cdots,M; \tag{13.1}$$

$$\sum_{k=r}^{R-1}\gamma_k(x_{k+1}^e - x_r^e) + \gamma_R \geq c_r\left(1 + \sum_{k=r}^{R-1}\rho_k(x_{k+1}^e - x_r^e) + \rho_R - \ln(c_r)\right), \qquad r = 1,\cdots,R; \tag{13.2}$$

$$\sum_{k=m}^{R-1}\gamma_k(x_{k+1}^e - x_m^e) + \gamma_R = 1; \tag{13.3}$$

$$\sum_{k=m}^{R-1}\rho_k(x_{k+1}^e - x_m^e) + \rho_R = 0; \tag{13.4}$$

$$\gamma_r \geq 0, \qquad r = 1,\cdots,R; \tag{13.5}$$

$$\rho_r \geq 0, \qquad t = 1,\cdots,R-1. \tag{13.6}$$

A sufficient condition for DSD efficiency is that some feasible solution γ_r^*, $r = 1,\cdots,R$; ρ_r^*, $r = 1,\cdots,R$, *exhibits a log-convex pattern*:

$$\ln\left(\sum_{k=r}^{R-1}\gamma_k^*(x_{k+1}^e - x_r^e) + \gamma_R^*\right) = \sum_{k=r}^{R-1}\rho_k^*(x_{k+1}^e - x_r^e) + \rho_R^*, \quad r = 1,\cdots,m-1,m+1,\cdots,R. \tag{14}$$

DSD efficiency (5), in contrast to DSD optimality (3), is defined in terms of marginal utility levels rather than utility levels, and hence (13.1) does not include squared outcome levels, in contrast to (11.1). Both expressions are however derived from the same piecewise-quadratic utility function (8.1) that is linear in parameters.

To prove the necessary condition, suppose that the evaluated prospect is the optimum for $u \in \mathcal{U}_3^*$, and hence also for the normalized function $v = u/u'(x_m^e)$. A solution to system (13) then is $\rho_r = a(\bar{x}_r) - a(\bar{x}_{r+1})$, $r = 1,\cdots,R-2$, and $\rho_{R-1} = a(\bar{x}_{R-1})$, for some tangency points $\bar{x}_r \in [x_r^e, x_{r+1}^e]$, $r = 1,\cdots,R-1$; $\rho_R = \ell(x_R^e)$; $\gamma_r = v''(\bar{\bar{x}}_{r+1}) - v''(\bar{\bar{x}}_r)$, $r = 1,\cdots,R-2$, and $\gamma_{R-1} = -v''(\bar{\bar{x}}_{R-1})$, for some tangency points $\bar{\bar{x}}_r \in [x_r^e, x_{r+1}^e]$, $r = 1,\cdots,R-1$; and $\gamma_R = v'(x_R^e)$.

The system (11)/(13) simultaneously imposes the constraints for convex log marginal utility (7.1)-(7.3) and the constraints for decreasing and convex marginal utility (8). Dropping the 'DARA constraints' (11.2)/(13.2), (11.4)/(13.4) and (11.6)/(13.6), while maintaining the other constraints, yields necessary and sufficient conditions for TSD optimality/efficiency.

Testing and diagnosis

We can specify linear programs to test the system of inequalities (11)/(13) for optimality/efficiency. The specific formulation would of course depend on the specific application area and decision problem. Our empirical section will develop a linear program for testing DSD efficiency of a stock market index relative to all portfolios formed from a set of base assets. An alternative approach is to include the linear inequalities in general method of moments estimation, along the lines of Post and Versijp (2007). This approach however requires convex quadratic programming (QP) and is computationally more demanding, especially for simulation and re-sampling methods.

Since the system (11)/(13) gives necessary conditions for *any* specification of the initial estimates c_t, $t = 1, \cdots, T$, failure to find a feasible solution directly implies that the evaluated prospect is DSD non-optimal/inefficient. By contrast, if we succeed to find a feasible solution, then we have to test for log-convexity before we can draw a conclusion. For this purpose, we may solve linear system (12)/(14) or, alternatively, visually inspect a graph with a logarithmic scale. If a log-convex pattern is found, then it follows that the evaluated prospect is DSD optimal/efficient. However, if log-convexity is violated, then further analysis is required.

Since violations of DARA stem from approximation error for the exponentiation of log marginal utility, we recommend changing the frame function if material deviations from log-convexity occur. For example, if a first-stage analysis based on a CRRA frame function identifies a feasible solution that violates DARA, a second-stage analysis could use a constant absolute risk aversion (CARA) frame function or, alternatively, use the optimal first-stage solution, ρ_t^*, to set $c_t = \exp(\sum_{k=t}^{T-1} \rho_k^* (z_{k+1} - z_t) + \rho_T^*)$, $t = 1, \cdots, T$.

A natural way to gauge the strength of our DSD optimality/efficiency test is to compare its results with those of the associated TSD test (which excludes the DARA constraints but maintains the prudence constraints). Any differences between the two sets of results would be fully attributable to violations of DARA by the TSD test. In our experience with applications in practical portfolio construction and empirical asset pricing, we generally find substantially more power for DSD tests than for TSD tests (particularly for skewed distributions with large differences in the means) and no material violations of DARA (using the CRRA frame function).

Numerical example

To illustrate the goodness of our local linear approximation, consider a simple investment example with five equally likely scenarios with a wide range of gross percentage investment returns x ranging from 60 (a 40% loss) to 180 (an 80% gain). We consider the CARA exponential marginal utility function $u'(x) \coloneqq \eta_1 \exp(-\theta_1 x), \eta_1 > 0, \theta_1 > 0$, and the CRRA power marginal utility function $v'(x) \coloneqq \eta_2 x^{-\theta_2}, \eta_2 > 0, \theta_2 > 0$. We select the risk aversion parameters (θ_1 and θ_2) to rationalize the average net return level: $\mathbb{E}[u'(x)(x - 100)] = \mathbb{E}[v'(x)(x - 100)] = 0$ and select the scalars (η_1 and η_2) to yield a median value of unity: $u'(x_3) = v'(x_3) = 1$, just as in our proposed procedure. We then approximate the CARA marginal utility levels using a local first-order approximation of the exponentiation around the CRRA log marginal return levels: $\widehat{u'(x)} \coloneqq v'(x) + v'(x)\left(\ln\big(u'(x)\big) - \ln\big(v'(x)\big)\right)$.

Table I and Figure 1 illustrate that the local linear approximation is very precise in this case, despite the wide range of outcomes and important differences between the two functional forms. In the graphs of Figure 1, the open dots represent the predicted (normalized) CARA marginal utility levels using a linear approximation (dotted lines) around CRRA marginal utility (the open diamonds). The largest error over the entire return range is about minus 2.5 percent of the relevant marginal utility level. In addition, the approximation errors results in only miniscule violations of DARA, witness the log marginal utility levels in the bottom right panel. These results are particularly encouraging because the relative outcome range in many applications is smaller than in this example and because most DARA functions are more robust to approximation using a CRRA frame function than the extreme case of exponential utility. The high precision can be explained by the fact that the exponential function and the CRRA frame function have the same normalization, overall level of risk aversion and general pattern, and are used to evaluate the same return levels, just as in our proposed procedure.

[Insert Table I about here]

[Insert Figure 1 about here]

5. Empirical application

We will now analyze the efficiency of a broad stock market portfolio using the DSD rule and other decision criteria. In this application, marginal utility can be interpreted as a pricing kernel and the violations of the first-order conditions as pricing errors or 'alphas'. Our analysis can be viewed as an empirical test for capital market equilibrium with a representative investor who holds the aggregated market portfolio. The analysis can also be interpreted as a revealed preference test for the observed behavior of investors who adopt a passive strategy of broad diversification. Finally, empirical evidence about which market segments and active strategies outperform a passive strategy is useful for active money managers.

Data set

Our market portfolio is a value-weighted average of all NYSE, AMEX and NASDAQ stocks. It is compared with ten benchmark stock portfolios that are formed, and annually rebalanced, based on individual stocks' market capitalization of equity (ME, or 'size'), and the one-month US Treasury bill. We use data on monthly value-weighted portfolio returns from July 1926 to December 2012 obtained from the data library of Kenneth French. These data are based on survivor bias-free historical stock market data from the Center for Research in Security Prices (CRSP) at the Booth School of Business at the University of Chicago. The size portfolios are of particular interest because a wealth of empirical research, starting with Banz (1981), suggests that small-cap stocks earn a return premium that seems to defy rational explanation.

We analyze gross holding period returns (HPRs) for all non-overlapping periods of H = 1, 3, 6 and 12 sequential months. We do not cover multi-year returns because the number of available non-overlapping multi-year return intervals seems too small and the results become sensitive to the specification of the starting year. In addition, our single-period optimization model seems not appropriate for a long-term investor who periodically adjusts her asset allocation. To analyze long-term returns, the use of simulated multi-year returns and a dynamic programming model may be more appropriate.

Table II shows descriptive statistics for the HPRs of the relevant portfolios and horizons. Not surprisingly, small-cap stocks tend to have a higher average return and standard deviation than large-cap stocks. Interestingly, the diversified market portfolio has a lower skewness than most of the concentrated benchmark portfolios. Apparently, broad diversification yields a relatively small reduction in downside risk at the cost of a relatively large reduction in upside potential, consistent with the observations of Simkowitz and Beedles (1978).

We treat the one-month T-bill as a risky asset, because it introduces potential inflation risk for investors who care about purchasing power and reinvestment risk for those who have a multi-month horizon. However, our results and conclusions are not materially affected by treating the bill as a riskless asset by analyzing returns in excess of the T-bill rate and/or using a bill with maturity equal to the assumed investment horizons of $H = 1,3,6,12$ months. For example, an analysis of one-year stock returns ($H = 12$) in excess of the one-year yield leads to very similar pricing errors and p-values as an analysis of one-year nominal stock returns and rolling over 12 consecutive one-month bills. This robustness is not surprising given the relatively low historical variation of the T-bill rates and the relatively small yield spread between one-year and one-month bills.

[Insert Table II about here]

Linear Program

To test whether the market portfolio is DSD efficient, we will design a linear program for the system of inequalities (13). In this application, the individual prospects are the $M = 10$ risky stock portfolios and a Treasury bill with return $\tilde{x}_F$, and the evaluated prospect is our market portfolio. The $R = 1{,}038/H$ time-series return observations ($H = 1, 3, 6, 12$) are interpreted as scenarios with equal probabilities $p_r = R^{-1}, r = 1, \cdots, R$. We will use the following LP problem:

$$\theta^* = \min_{\gamma_r, \rho_r, \theta} \theta \qquad (15.1)$$

$$\text{s.t.} \sum_{r=1}^{R} \left(\sum_{k=r}^{R-1} \gamma_r (x_{k+1}^e - x_r^e) + \gamma_R \right) \left(x_r^e - x_{j,r} \right) p_r + \theta \geq 0, \qquad j = 1, \cdots, M; \qquad (15.2)$$

$$\sum_{r=1}^{R}\left(\sum_{k=r}^{R-1}\gamma_r(x_{k+1}^e - x_r^e) + \gamma_R\right)\left(x_r^e - x_{F,r}\right)p_r = 0; \tag{15.3}$$

$$\sum_{k=r}^{R-1}\gamma_r(x_{k+1}^e - x_r^e) + \gamma_R \geq c_r\left(1 + \sum_{k=r}^{R-1}\rho_k(x_{k+1}^e - x_r^e) + \rho_R - \ln(c_r)\right), \qquad r = 1,\cdots,R; \tag{15.4}$$

$$\sum_{k=m}^{R-1}\gamma_k(x_{k+1}^e - x_m^e) + \gamma_R = 1; \tag{15.5}$$

$$\sum_{k=m}^{R-1}\rho_k(x_{k+1}^e - x_m^e) + \rho_R = 0; \tag{15.6}$$

$$\gamma_r \geq 0, \qquad r = 1,\cdots,R; \tag{15.7}$$

$$\rho_r \geq 0, \qquad r = 1,\cdots,R-1. \tag{15.8}$$

The objective function is the parameter θ, the largest positive pricing error of the 10 size portfolios. Restriction (15.2) bounds the alphas from above by this parameter. A value of $\theta^* = 0$ is required to classify the market portfolio as efficient; a value of $\theta^* > 0$ implies inefficiency. We do not explicitly impose the restriction $\theta \geq 0$, because, by construction, at least one of the 10 portfolios must have a non-negative alpha.

Alternative specifications of the objective function include minimizing a weighted sum of squared alphas (which requires convex QP) or absolute alphas (which leads to LP but requires additional variables and constraints) and minimizing the largest absolute alpha. Our results are robust to the use of these alternative objective functions, presumably because the data set is dominated by the large positive alpha of small-cap stocks.

Restriction (15.3) requires the pricing kernel to be consistent with the equity premium by requiring the pricing error of the T-bill to be zero ($\mathbb{E}[u'(\tilde{x}^e)(\tilde{x}^e - \tilde{x}_F)] = 0$), a standard assumption in the asset pricing literature. Whereas we normalize median marginal utility to unity (15.5) and median log marginal utility to zero (15.6), the convention in the asset pricing literature is to set the *average* value of the pricing kernel equal to unity; we therefore rescale the optimal marginal utility levels after the estimation.

Our LP problem assumes that the portfolio possibilities consist of convex combinations of the ten size portfolios and the T-bill. One could generalize this assumption to allow for a more general portfolio set by replacing the size portfolios with extreme portfolios that include short positions. However, the market portfolio assigns strictly positive weight to all stocks and hence short-sales restrictions are not binding and do not affect our results and conclusions.

The number of variables and constraints is linear in the number of base assets and time-series observations, and the linear program is relatively small (given the current state of computer hardware and solver software) for our datasets. We were able to perform extensive simulation and bootstrapping exercises at relative ease using the LP module of SAS ran on a Levono ThinkPad T530i with a 2.4GHz, 16GB DDR3 Intel Pentium CPU.

Apart from the DSD efficiency test, we also apply a TSD efficiency test that drops the DARA constraints and Post's (2003) SSD efficiency test based on general piecewise-linear utility (without requiring prudence or DARA). Finally, we apply an M-V efficiency test that assumes a linear marginal utility function with coefficients based on the equity premium (restriction (15.3)) and an average value of unity, so that the pricing errors amount to Jensen's (1968) alphas.

We selected the prior estimates $c_r, r = 1, \cdots, R,$ using a (normalized) power utility function that is calibrated to the historical equity premium, that is, $c_r = \eta(x_r^e)^{-\theta}$, $r = 1, \cdots, R$, with $\eta > 0$ and $\theta > 0$ such that $\sum_{r=1}^{R} c_r p_r = 1$ and $\sum_{r=1}^{R} c_r \left(x_r^e - x_{F,r}\right) p_r = 0$. Our results and conclusions are not materially affected by using other frame functions forms, such as an exponential function, provided the same normalization and general risk aversion level are used.

Bootstrap method

Portfolio efficiency tests are well-established for the multivariate normal distribution. Gibbons, Ross and Shanken (1989) develop test statistics for mean-variance efficiency with a known sampling distribution. Levy and Roll (2010) propose an interesting reverse-engineering approach to find the smallest perturbations to the mean vector and covariance matrix that rationalize a given portfolio. It is generally difficult to apply similar approaches to SD rules, which do not specify a parametric functional form for the return distribution. Instead, re-sampling methods have emerged as the dominant method for statistical inference on SD relations (Barrett and Donald (2003); Linton *et al.* (2005); and Linton *et al.* (2013)).

In our analysis, we will use a re-centered IID bootstrap approach that repeatedly applies LP test (15) to pseudo-samples drawn from the original sample. Under the assumption of serial IIDness, the empirical return distribution is a consistent estimator of the population return distribution, and bootstrap pseudo-samples can simply be obtained by random sampling with replacement from the empirical return distribution.

For dynamic stochastic processes, a block bootstrap may be more appropriate, but our samples are relatively small (particularly the sample of annual returns) and GARCH effects are limited for low-frequency returns to diversified stock portfolios. The IID bootstrap can often be used for conservative statistical inference for non-IID processes, and a block bootstrap is likely to further lower the p-values in our analysis and increase the evidence against market portfolio efficiency. For example, applying a block bootstrap with a block size of 12 months to our data set of monthly returns ($H = 1$) leads to a slightly smaller p-value for the DSD efficiency test than the IID bootstrap (0.051 vs. 0.073).

To ensure that the bootstrap process obeys the null hypothesis of market portfolio efficiency, we first re-center the empirical distribution in the spirit of Hall and Horowitz (1996). We correct the original time-series of returns for a given base asset $j = 1, \cdots, 10$, by subtracting the estimated pricing error $\hat{\alpha}_j \coloneqq R^{-1} \sum_{r=1}^{R} u'(x_r^e)\left(x_r^e - x_{j,r}\right)$ from every return observation to obtain re-centered observations: $\hat{x}_{j,r} \coloneqq x_{j,r} - \hat{\alpha}_j \;\; r = 1, \cdots, R$. While this adjustment aligns the assets' means with the null hypothesis, it does not affect the general risk levels and the dependence structure between the assets.

We implement the bootstrap by generating 10,000 pseudo-samples of the same size as the original sample through random draws with replacement from the re-centered original sample, and test market portfolio efficiency in every pseudo-sample. Finally, we compute the critical values for the original test statistics from the percentiles of the bootstrap distribution. We performed extensive simulations to verify that our bootstrap procedure yields the correct statistical size and more statistical power than asymptotic inference methods using the return generating process of Post and Versijp (2007).

Empirical results

Table III summarizes the test results (test statistic, bootstrap p-value and pricing errors) for the various decision criteria (M-V, SSD, TSD, DSD) and return intervals (H=1, 3, 6, 12). Figure 2 displays the pricing kernels, using a logarithmic scale in order to emphasize possible violations of DARA (or log-convexity of the kernel).

M-V efficiency cannot be rejected in a convincing way at conventional significance levels, consistent with the results of Gibbons, Ross and Shanken (1989), Levy and Roll (2010), among others. For monthly returns and annual returns, the M-V alpha of the small-cap portfolio ME1 is about three percent per annum and the bootstrap p-value hovers around ten percent. For quarterly and semi-annual returns, the alphas are smaller, around one percent per annum, and not statistically significant.

One unappealing feature of the M-V kernel is that it sometimes takes negative values for the largest market returns, placing a penalty on outperforming the market during market upswings. This feature illustrates that the M-V criterion generally is not consistent with the conditions of non-satiation and no-arbitrage (as discussed by, for example, Borch (1969) and Dybvig and Ingersoll (1982)). In addition, the M-V kernel is linear (IARA) and hence not log-convex (DARA). The kernel does not reward positive skewness, and therefore seems to underestimate the appeal of small-cap stocks. The violations of non-satiation and DARA occur both in the original samples (which determine the value of the test statistic) and in the bootstrap pseudo-samples (which determine the bootstrap p-value).

The SSD efficiency test yields substantially smaller pricing errors and higher p-values than the M-V efficiency test for all horizons. Although the SSD criterion avoids negative values for the kernel, it imposes no structure beyond non-satiation and risk aversion. Not surprisingly, the SSD kernel generally does not resemble a well-behaved marginal utility function. Notably, the SSD kernel generally takes the shape of a step function with large concave segments, and it penalizes small-cap stocks for having a relatively high positive skewness, in violation of prudence.

The TSD criterion imposes prudence and avoids concave segments of the kernel. The TSD results are remarkably similar to the M-V results. The TSD kernel tends to be linear (if the M-V kernel is globally non-negative) or two-piece linear with a single kink to avoid negative values for the largest positive market returns. In case of a two-piece linear shape, the TSD kernel assigns a larger (positive) weight to scenarios with large positive market returns, increasing the small-cap pricing errors compared with the M-V alphas. The TSD kernel is convex but not log-convex and hence violates DARA. Although it avoids penalizing skewness, the TSD kernel generally does not reward skewness, and therefore seems to underestimate the appeal of small-cap stocks. Very similar results are obtained using the fourth-order SD criterion (not reported here). This finding is not surprising, given that the general N-th order SD criterion allows for quadratic utility and IARA.

The DSD criterion leads to a material increase in the pricing errors relative to the M-V and TSD criteria for every return interval and the most convincing rejections of market portfolio efficiency. Most notably, for annual returns, the DSD criterion increases the small-cap alpha by more than 100 basis points from about three percent to four percent per annum, and the p-value drops from ten to five percent. The DSD test does not allow marginal utility to be negative or locally linear and it penalizes the market portfolio for offering less skewness than small-cap stocks. It seems that the appeal of small-cap stocks is substantially stronger and the market portfolio is substantially less efficient than the M-V efficiency and N-th order SD efficiency tests suggest.

The optimal DSD kernel is very similar to exponential marginal utility (CARA), a boundary case of DARA.[5] This result is not related to our linear approximation of the exponentiation of log marginal utility, which is based on a CRRA frame function and, in addition, is too accurate to produce non-trivial bias for the relevant return range. The tendency to CARA instead reflects the discriminating power of the DARA assumption in this application. Given the assumption of market portfolio efficiency, the returns data set supports IARA: the highest-yielding assets also have the highest positive skewness. Hence, imposing DARA increases the evidence against market portfolio efficiency. The DSD pricing errors are smallest for the boundary case of CARA. Assuming strictly

[5] A notable deviation from CARA occurs for $H = 6$. The DSD kernel drops steeply from 6.84 for $x = 50.71\%$ (the market HPR in 1931H2) to 1.56 for $x = 54.10\%$ (the HPR in 1932H1), before decreasing at a moderate exponential rate for the remaining 169 observations. This pattern implies that the relative risk aversion (RRA) quotient decreases in this return range. The large drop occurs because small-caps underperform the market in 1931H2 but outperform in 1932H1. The kernel is clearly over-fitted to the sparse data in the left tail. One could obtain more robust results by using some form of kernel smoothing, for example, capping the ARA decrements between subsequent (ranked) observations. An alternative approach is to impose IRRA using the method described in the Appendix. These approaches would increase the DSD alphas for small-caps for $H = 6$ and do not affect our overall results and conclusions.

decreasing ARA, for example, using a power utility function, tends to further increase the empirical evidence against market portfolio efficiency.

We caution against interpreting our results as evidence in favor of exponential utility and also against extrapolating the functional form outside the observed return range. It is well known that highly risk-averse exponential functions lead to paradoxes such as the rejection of Markowitz' 50/50 gamble between 'breaking even' and a 'blank check'. A more balanced interpretation is that a function that resembles the exponential for the observed range (but not necessarily for the range of 'blank checks') gives the lowest pricing errors in the class of DARA functions, and other DARA functions (some of which are arguably more realistic) produce even larger pricing errors in our application. Finally, for multi-year horizons (not reported here) the optimal DSD kernel deviates from the boundary case and displays strictly DARA.

[Insert Table III about here]

[Insert Figure 2 about here]

6. Concluding remarks

Several conclusions can be drawn from our empirical application. First, the DSD criterion has substantially more discriminating power than the TSD criterion. Vickson (1975b) demonstrates that the TSD and DSD criteria are equivalent when the prospects have equal means. However, the average returns of financial assets show substantial variation due to risk premiums and/or pricing errors, and the two decision criteria may diverge in asset pricing and asset allocation applications. In an empirical application to stock portfolios, Vickson and Altmann (1977) show that the pairwise DSD test is only slightly more powerful than the pairwise TSD test because both tests suffer from data sparsity in the left tail of the return distribution. However, our convex dominance tests are more robust to the left-tail problem, and hence more powerful, than pairwise dominance tests.

In our analysis, the TSD rule (and every higher-order SD rule) does not differ materially from the basic M-V criterion. The M-V and TSD criteria obey prudence but violate DARA by allowing marginal utility to be (globally or locally) linear. Both criteria substantially underestimate the degree of inefficiency of the stock market index and the appeal of concentrated portfolios of small-cap stocks to DARA investors. The TSD criterion does allow for DARA utility function, but in our analysis, it assumes an IARA shape in order to lower the alpha for small-cap stocks, which have a relatively high mean and a relatively high positive skewness.

These results are consistent with the observation of Levy and Markowitz (1979) that the M-V approximation does not work well for exponential utility (CARA) with a high level of risk aversion: in our analysis, the optimal DSD utility function approximates the exponential function and, in addition, the assumed level of risk aversion is high in order to rationalize the historical equity premium. Our findings are also consistent with the conclusion by Basso and Pianca (1997) that the DARA rule improves the stochastic dominance criteria of any order for determining option pricing bounds.

Second, our linear approximation to the DSD criterion appears very accurate. Formally, our tests represent necessary but not sufficient conditions for optimality/efficiency. However, our optimal marginal utility functions generally show no or minimal violations of DARA, which implies that the approximation is perfect or very good. Indeed, our results and conclusions are not materially affected by using a second-order approximation rather than a first-order approximation to the exponentiation of log marginal utility (results available upon request).

We attribute the strength of our tests to the joint normalization of marginal utility and log marginal utility and the use of prior parametric estimates for log marginal utility. By standardizing marginal utility at the median outcome level, we are able to apply the corresponding normalization to log marginal utility, avoiding possible inconsistencies in the location of the two functions. By calibrating a 'frame function' to the decision problem, we obtain good prior estimates with the same normalization, general level of risk aversion and general properties (positive, decreasing and log-convex) as DSD marginal utility, allowing for a local linear approximation to the exponentiation of log marginal utility.

Third, the DSD criterion can now be applied in general multivariate cases, including the comparison of a given prospect with a polyhedral set of linear combinations of alternatives. DSD can be implemented in such cases by solving a relatively small system of linear inequalities (13) by means of LP. The problem size is sufficiently small to allow for a re-sampling approach to statistical inference without excessive CPU time. Another class of multivariate applications compares a given prospect with a discrete set of alternative prospects (without allowing for mixtures) and can also be implemented by solving a system of linear inequalities (11). A generalization of our tests based on a second-order approximation to the exponentiation can be implemented using convex QP.

Our analysis can be extended to decision rules based on increasing relative risk aversion (IRRA) using piecewise-power utility functions. The appendix provides further details. Although IRRA is a common assumption in utility theory, it appears less powerful than DARA in our experience, because it allows for quadratic utility (IARA ⇒ IRRA) and violations of prudence. As a case in point, in our empirical analysis, the (approximately) linear M-V/TSD kernel and (approximately) exponential DSD kernel show no material violations of IRRA and imposing IRRA therefore has a limited effect.

Finally, we hope that this study will contribute to the further proliferation of the SD methodology by improving its discriminating power and reducing its computational burden in practically relevant applications.

Appendix

This section discusses extensions of our framework to impose increasing relative risk aversion (IRRA). Let $r(x) := a(x)x = -u''(x)x/u'(x)$ represent the Arrow-Pratt relative risk aversion (RRA) quotient. We assume strictly positive outcomes ($y_1 > 0$), which may require an additive data transformation (for example, using total wealth or gross returns). Consider the following set of IRRA functions:

$$\mathcal{U}_2^* := \{u \in \mathcal{U}_2 : u'(x) > 0; r'(x) \geq 0 \quad \forall x \in \mathcal{D}\}. \tag{A.1}$$

These functions are a subset of the SSD functions $\mathcal{U}_2$ rather than the TSD functions $\mathcal{U}_3$, because IRRA (in contrast to DARA) does not imply prudence. Using $\mathcal{U}_2^*$ in Definition 1 and 2 yields definitions of 'IRRA SD' (ISD) optimality and efficiency. We can formulate IRRA in terms of the composite function $\mathcal{A}(\ln(x)) := -\ln\big(u'(x)\big)$, which is constructed to yield $(\mathcal{A}(\ln(x)))' = \mathcal{A}'(\ln(x))\frac{1}{x} = a(x)$ and $\mathcal{A}'(\ln(x)) = r(x)$. IRRA amounts to

$$r'(x) \geq 0 \Leftrightarrow 0 \leq \frac{\partial \mathcal{A}'(\ln(x))}{\partial x} = \frac{\partial \mathcal{A}'(\ln(x))}{\partial \ln(x)} \frac{\partial \ln(x)}{\partial x} = \mathcal{A}''(\ln(x)) \frac{1}{x} \Leftrightarrow \mathcal{A}''(\ln(x)) \geq 0. \tag{A.2}$$

PROPOSITION 4 (EXPONENTIATION)

$$\mathcal{U}_2^* = \left\{u \in \mathcal{U}_2 : u'(x) = \exp\left(-\mathcal{A}(\ln(x))\right) \ - \mathcal{A} \in \mathcal{U}_1^* \quad \forall x \in \mathcal{D}\right\}; \tag{A.3}$$

$$\mathcal{U}_1^* = \{u \in \mathcal{C}^2 : u'(x) \geq 0, u''(x) \geq 0 \quad \forall \exp(x) \in \mathcal{D}\}. \tag{A.4}$$

The ISD criterion thus requires that log marginal utility $\ell(x) \coloneqq \ln(u'(x))$ is a decreasing concave function of the log outcomes.

PROPOSITION 5 (ISD LOG MARGINAL UTILITY) *For any (normalized) utility function $u \in \mathcal{U}_2^*$ and a discrete set of outcomes $z_1 \leq \cdots \leq z_T$, we can represent the levels of log marginal utility by the corresponding levels of a decreasing and concave piecewise-linear function of the log outcome levels:*

$$\ell(z_t) = \varrho_0 + \sum_{k=1}^{t-1} \varrho_k \ln(z_k/z_t), \qquad t = 1, \cdots, m-1, m+1, \cdots, T; \tag{A.5}$$

$$\ell(z_m) = \varrho_0 + \sum_{k=1}^{m-1} \varrho_k \ln(z_k/z_m) = 0; \tag{A.6}$$

$$\varrho_t \geq 0, \qquad t = 1, \cdots, T-1. \tag{A.7}$$

In this proposition, $\varrho_0 = \ell(z_1)$, $\varrho_1 = r(\underline{z}_1)$, and $\varrho_t = r(\underline{z}_t) - r(\underline{z}_{t-1})$, $t = 2, \cdots, T-1$, for some tangency points $\underline{z}_t \in [z_t, z_{t+1}]$, $t = 1, \cdots, T-1$.

The log-log structure implies that the ISD criterion can be represented by piecewise-power functions:

$$u(z_t) = \begin{cases} -\left(1 - \sum_{k=1}^{t-1} \rho_k\right)^{-1} \exp\left(\rho_0 + \sum_{k=1}^{t-1} \rho_k \ln(z_k)\right) z_t^{\left(1 - \sum_{k=1}^{t-1} \rho_k\right)} + \varphi_t, & t = t_0, \ldots, T \\ \exp(\rho_0) z_t, & t = 1, \ldots, t_0 - 1; \end{cases} \tag{A.8}$$

$$t_0 := \min_t \left\{ t : \sum_{k=1}^{t} \rho_k > 0 \right\}. \tag{A.9}$$

The constants $\varphi_t, t = t_0, \ldots, T$, are selected to ensure continuity of the utility levels at the interval boundaries. If $\sum_{k=1}^{t-1} \rho_k = 1$ for some $t = i$ then (A.8) must be modified to:

$$u(z_i) = \exp\left(\varrho_0 + \sum_{k=1}^{i-1} \rho_k \ln(z_k) \right) \ln(z_i) + \varphi_i. \tag{A.8'}$$

Like the piecewise-exponential DSD functions, these piecewise-power ISD functions are non-linear and non-convex in the parameters and hence unpractical in our applications. However, we can linearize $u \in \mathcal{U}_2$ and $u'(x) = \exp(\ell(x))$ by analogy to our approach to DSD.

PROPOSITION 6 (SSD UTILITY) *For any (normalized) utility function $u \in \mathcal{U}_2$, and a discrete set of outcomes $z_1 \leq \cdots \leq z_T$, we can represent the levels of utility (marginal utility) by the corresponding levels of an increasing and concave piecewise-linear (decreasing piecewise-constant) function of the outcome levels:*

$$u(z_t) = \sum_{k=t}^{T-1} \gamma_k (z_t - z_{k+1}) + \gamma_T, \qquad t = 1, \cdots, T; \tag{A.10}$$

$$u'(z_t) = \sum_{k=t}^{T-1} \gamma_k, \qquad t = 1, \cdots, m-1; m+1, \cdots, T; \tag{A.11}$$

$$u'(z_m) = \sum_{k=m}^{T-1} \gamma_k = 1; \tag{A.12}$$

$$\gamma_t \geq 0, \qquad t = 1, \cdots, T-1. \tag{A.13}$$

The needed adjustments to Theorem 1 and 2 now follow from Proposition 4, Proposition 5, Proposition 6 and Lemma 1.

To impose the attractive combination of DARA and IRRA, we could simply impose the two sets of restrictions simultaneously. This approach however involves several redundant variables and constraints. A more computationally efficient approach is to complement the DSD restrictions with (A.7) and

$$\sum_{k=t}^{T-1} \rho_k (z_{k+1} - z_t) + \rho_T = \varrho_0 + \sum_{k=1}^{t-1} \varrho_k \ln(z_k / z_t) \,, \qquad t = 1, \cdots, T. \tag{A.14}$$

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Table I Numerical Example

We consider a simple investment example with five scenarios ($t = 1, \cdots, 5$) with equal probability ($p_t = 0.2$) and a gross percentage investment returns x ranging 60 (a 40% loss) to 180 (an 80% gain). We consider two different marginal utility functions: $u'(x) = 4.068\exp(-0.012x)$ (CARA) and $v'(x) = 335.35x^{-1.215}$ (CRRA). We approximate the CARA marginal utility levels using a local first-order approximation of the exponential transformation around the CRRA log marginal return levels: $\widehat{u'(x)} \coloneqq v'(x) + v'(x)\left(\ln\big(u'(x)\big) - \ln\big(v'(x)\big)\right)$. The last two columns show the error $\widehat{u'(x)} - u'(x)$ and percentage error $\widehat{u'(x)}/u'(x) - 1$.

Scenario (t)	Prob. (p_t)	Gross Return (x_t)	CRRA $v'(x)$	CARA $u'(x)$	Pred. $\widehat{u'(x)}$	Error	%Error
1	0.2	60	2.321	2.017	1.995	-0.022	-1.1%
2	0.2	90	1.418	1.420	1.420	0.000	0.0%
3	0.2	120	1.000	1.000	1.000	0.000	0.0%
4	0.2	150	0.763	0.704	0.702	-0.002	-0.3%
5	0.2	180	0.611	0.496	0.483	-0.012	-2.5%

Table II Descriptive Statistics

The table shows descriptive statistics for gross holding-period returns to ten benchmark stock portfolios, the stock market portfolio and the one-month US Treasury bill. The benchmark portfolios are based on individual stocks' market capitalization of equity, and each represent a value-weighted average of a segment of the cross-section of stocks (using NYSE size break points). The stock market portfolio is a value-weighted average of all NYSE, AMEX and NASDAQ stocks. The sample period ranges from July 1926 to December 2012 (1,038 months). Separate statistics are shown for a return interval of H = 1, 3, 6 and 12 months. The raw month-end-to-month-end returns are taken from Kenneth French' data library. Multi-month HPRs are obtained as the product of the relevant gross monthly returns.

	H=1				H=3			
Portf	Mean	Stdev	Skew	Kurt	Mean	Stdev	Skew	Kurt
ME1 (S)	101.43	10.16	3.69	37.19	104.30	21.48	3.05	22.62
ME2	101.26	8.89	2.22	22.00	103.78	18.51	2.61	21.69
ME3	101.26	8.13	1.89	20.19	103.78	17.09	2.71	24.64
ME4	101.21	7.52	1.51	15.69	103.62	15.52	2.20	18.78
ME5	101.17	7.22	1.10	12.91	103.52	15.02	2.27	20.91
ME6	101.16	6.89	0.98	11.98	103.49	14.02	1.98	17.51
ME7	101.11	6.53	0.76	10.87	103.34	13.20	1.52	13.67
ME8	101.06	6.19	0.70	10.66	103.17	12.26	1.44	13.24
ME9	101.00	5.89	0.52	10.33	103.00	11.74	1.25	12.92
ME10 (L)	100.87	5.10	0.05	6.40	102.60	9.92	0.45	8.17
T-bill	100.29	0.25	1.04	1.26	100.87	0.75	0.97	0.91
Mkt	100.92	5.41	0.13	7.34	102.75	10.70	0.74	9.30
	H=6				H=12			
Portf	Mean	Stdev	Skew	Kurt	Mean	Stdev	Skew	Kurt
ME1 (S)	108.60	28.53	1.42	7.89	119.37	40.60	0.83	1.19
ME2	107.56	24.93	1.41	10.57	116.65	35.04	0.67	1.90
ME3	107.57	22.67	1.33	10.87	116.41	31.92	0.76	3.07
ME4	107.23	20.76	1.11	8.34	115.66	29.53	0.45	1.04
ME5	107.04	19.90	0.88	8.11	115.06	27.39	0.20	1.13
ME6	106.98	18.78	0.66	6.67	114.92	26.32	0.15	0.64
ME7	106.69	17.70	0.37	5.07	114.43	25.31	0.09	1.02
ME8	106.34	16.43	0.39	5.01	113.40	23.17	0.06	1.38
ME9	106.00	15.59	-0.17	4.52	112.74	21.90	-0.32	0.87
ME10 (L)	105.19	13.50	-0.51	2.84	110.92	19.23	-0.44	-0.07
T-bill	101.74	1.50	0.93	0.74	103.49	2.99	0.92	0.72
Mkt	105.50	14.45	-0.32	3.06	111.63	20.39	-0.42	0.02

Table III Portfolio Efficiency Tests

The table shows results for testing efficiency of the value-weighted market portfolio relative to the ten size portfolios and the one-month Treasury bill. Separate results are shown for the M-V, SSD, TSD, and DSD criteria and gross holding-period returns of H = 1, 3, 6 and 12 months. Statistical inference is based on a re-centered bootstrap that generates 10,000 pseudo-samples from the re-centered original sample of H-month returns. The original returns are re-centered by subtracting the assets' estimated pricing errors, so that the market portfolio becomes efficient. The pricing kernels are rescaled to an average value of 1 (after the estimation). For the sake of interpretation and comparability, the test statistics and pricing errors for an interval of H months are 'annualized' by multiplication with (12/H). Asterisks *, ** and *** indicate that the bootstrap p-value for a test statistic or pricing error is smaller than 10%, 5% and 1%, respectively.

		H=1				*H*=3			
		M-V	SSD	TSD	DSD	M-V	SSD	TSD	DSD
Goodness of fit	Test stat	2.889*	0.726	2.910*	3.247*	1.087	0.640	1.215	1.897
	P-value	0.082	0.447	0.080	0.073	0.394	0.343	0.375	0.232
Pricing errors	ME1 (S)	2.889*	0.692	2.910*	3.247*	1.020	-0.183	1.215	1.897
	ME2	1.159	-0.385	1.166	1.340	0.052	-0.616	0.184	0.627
	ME3	1.588	0.309	1.599	1.762	0.577	0.112	0.716	1.105
	ME4	1.539	0.726	1.553	1.694	0.871	0.234	0.978	1.311
	ME5	1.261	0.381	1.273	1.368	0.626	0.301	0.729	1.001
	ME6	1.390	0.671	1.404	1.509	1.087	0.640	1.178	1.429
	ME7	1.189	0.726	1.200	1.282	0.928	0.640	0.989	1.162
	ME8	0.844	0.652	0.851	0.924	0.809	0.640	0.857	0.999
	ME9	0.482	0.246	0.488	0.531	0.397	0.438	0.430	0.512
	ME10 (L)	-0.092	0.053	-0.093	-0.099	0.010	0.099	0.002	-0.036
		H=6				*H*=12			
		M-V	SSD	TSD	DSD	M-V	SSD	TSD	DSD
Goodness of fit	Test stat	1.324	0.566	1.397	1.962	3.050*	0.944	3.295*	4.055**
	P-value	0.336	0.396	0.324	0.198	0.098	0.230	0.099	0.050
Pricing errors	ME1 (S)	1.324	-0.004	1.397	1.962	3.050*	0.944	3.295*	4.055**
	ME2	0.346	-0.410	0.374	0.635	1.180	-0.091	1.275	1.777
	ME3	0.942	0.387	1.001	1.580	1.662	0.626	1.795	2.330
	ME4	1.122	0.246	1.185	1.779	1.541	0.422	1.665	2.102
	ME5	0.910	0.445	0.960	1.390	1.470	0.944	1.588	1.857
	ME6	1.207	0.566	1.266	1.734	1.561	0.615	1.686	2.031
	ME7	1.036	0.566	1.102	1.841	1.316	0.676	1.422	1.656
	ME8	0.904	0.537	0.952	1.433	1.093	0.767	1.181	1.359
	ME9	0.558	0.404	0.559	0.240	0.732	0.503	0.791	0.840
	ME10 (L)	-0.034	0.086	-0.041	-0.146	-0.115	-0.022	-0.124	-0.160

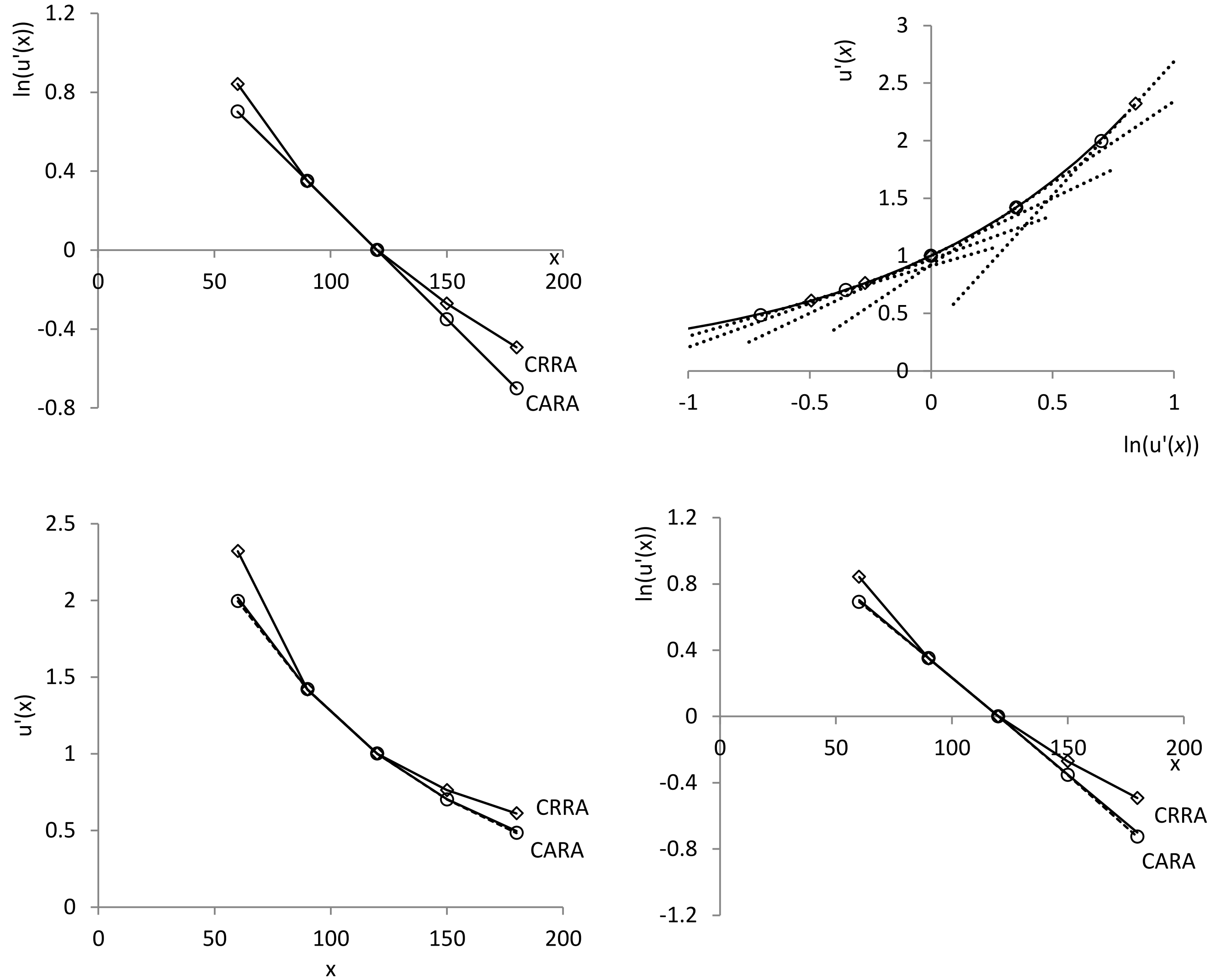

Figure 1 Numerical Example

We consider a simple investment example with five scenarios ($t = 1, \cdots, 5$) with equal probability ($p_t = 0.2$) and a gross percentage investment returns x shown in Table I. The graphs illustrate the approximation of the CARA marginal utility levels $u'(x) = 4.068\exp(-0.012x)$ using a local first-order approximation of the exponential transformation around the log of the CRRA marginal return levels $v'(x) = 335.35x^{-1.215}$: $\widehat{u'(x)} := v'(x) + v'(x)\left(\ln\left(u'(x)\right) - \ln\left(v'(x)\right)\right)$. The top left panel shows the logs of the two marginal utility functions, using open dots for the CARA function and open diamonds for the CRRA function. The top right panel shows the local first-order approximation of the exponential transformation, using dotted lines for the tangency lines at $\ln\left(v'(x)\right)$. The bottom left panel displays the two marginal utility functions together with the resulting approximation to the CARA marginal utility function (dotted line). The bottom right panel shows the natural logs of the two functions and the approximation.

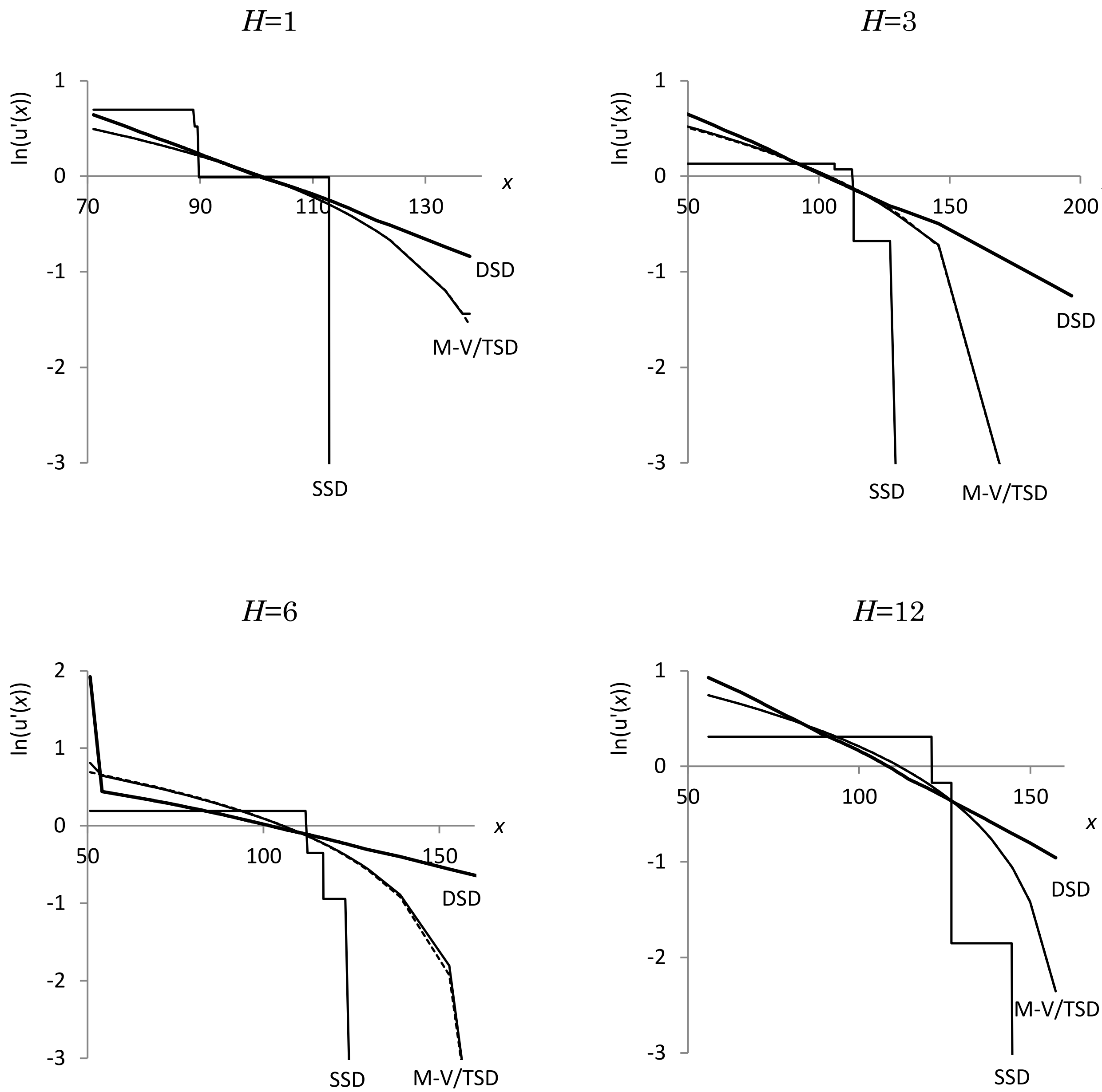

Figure 2 Log Pricing Kernels

The graphs show the natural logarithms of the pricing kernels generated by our tests for M-V, SSD, TSD and DSD efficiency of the value-weighted market portfolio relative to the ten size portfolios and the one-month Treasury bill. Separate results are shown for gross holding-period returns of $H = 1$, 3, 6 and 12 months. For the sake of comparison, the pricing kernels are rescaled to an average value of 1. In order to avoid overstretching the graph, the ordinate is truncated at a value of -3 (or a value of the pricing kernel of $\exp(-3) \approx 0.05$). The M-V kernel is represented by a dotted line in order to distinguish it from the TSD kernel, which is very similar. A utility function exhibits DARA if and only if log marginal utility is convex.